Returning to the Schaeffer model...

What happens to the bifurcation diagram if the differential equation is perturbed?

$$\frac{dN}{dt} = f(N; E)$$

$\underline{\underline{vs}}$

"imperfection"

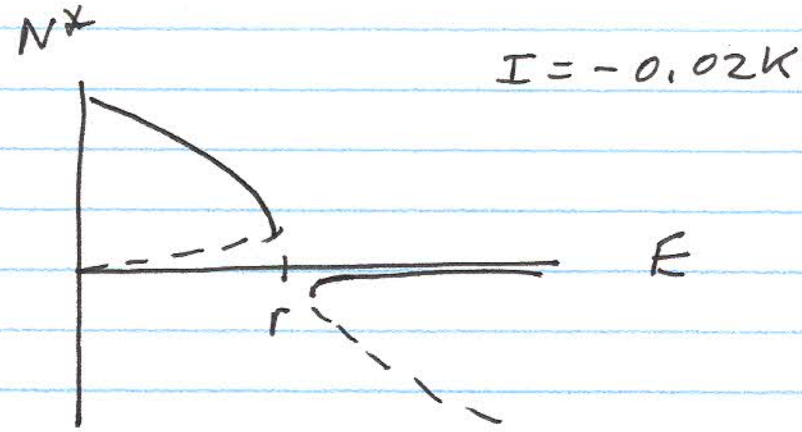
$$\frac{dN}{dt} = f(N; E) + \varepsilon\, g(N)$$

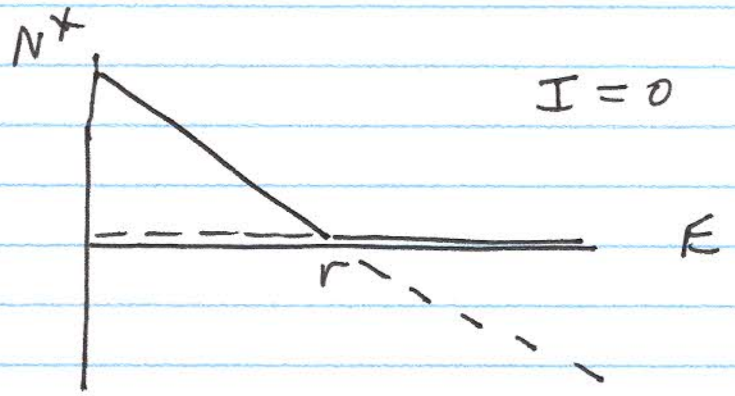
Suppose $\varepsilon$ small. Do both systems have similar bifurcation diagrams?

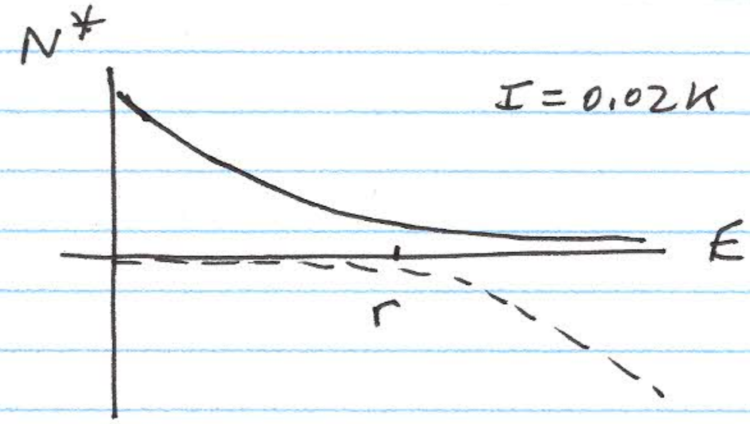
Suppose we add a little immigration/emigration to the Schaeffer model. What happens to the bifurcation diagram?

Is it robust to small changes?

$$\frac{dN}{dt} = rN\left(1 - \frac{N}{K}\right) - EN + I$$



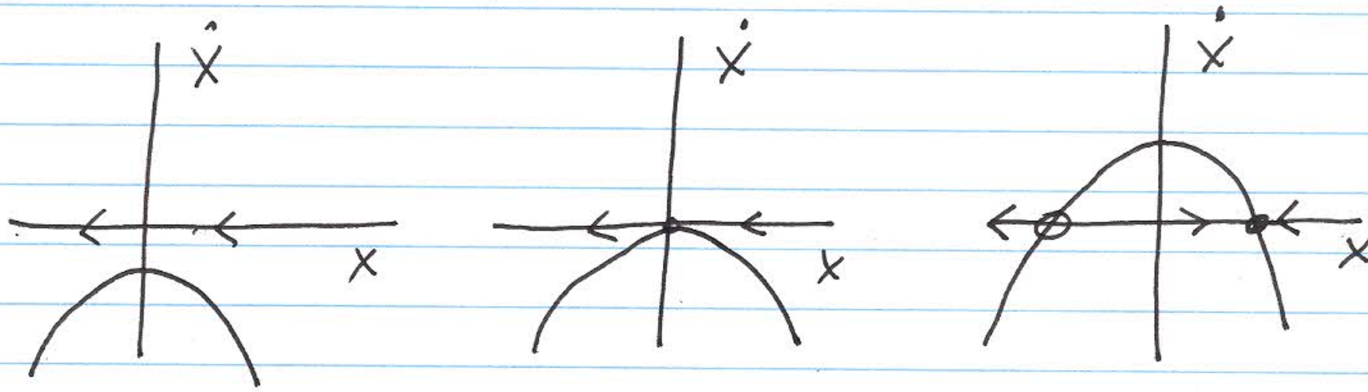

Bifurcation diagram is structurally unstable

see Kot page 23

# Elementary Bifurcations

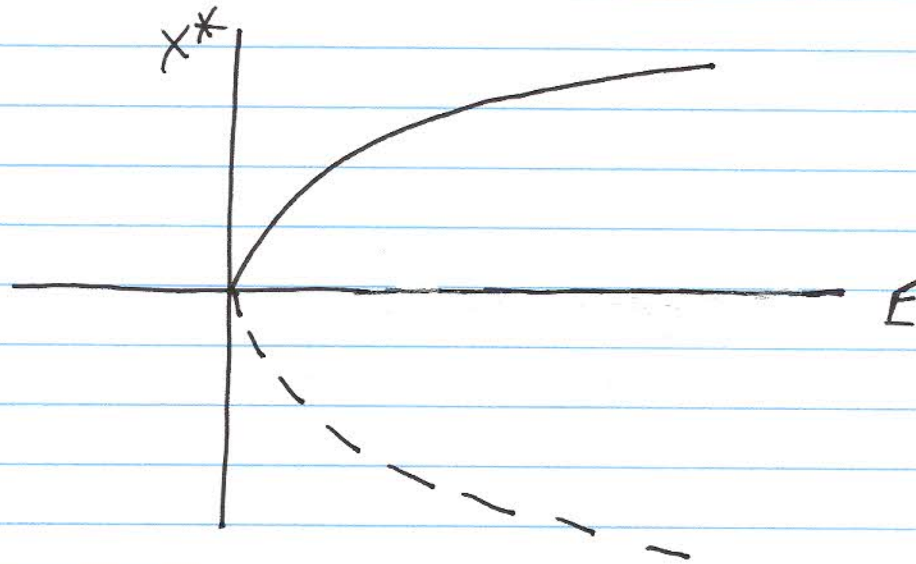
<u>Saddle-node bifurcation</u> - This is the basic mechanism by which fixed points are created or destroyed

$$\dot{x} = E - x^2 = f(x;E)$$




$E < 0$ $\quad$ $E = 0$

$$\frac{\partial f}{\partial x} = -2x$$



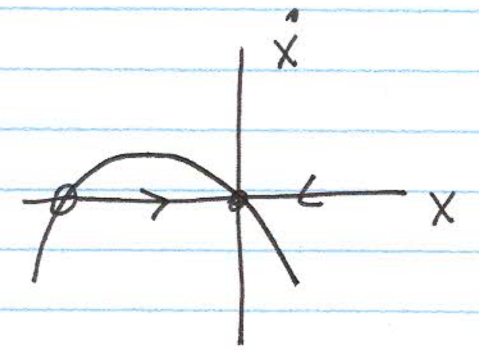

$$0 = f(x^*;E)$$
$$\Rightarrow \quad E = x^{*2}$$

Bifurcation at

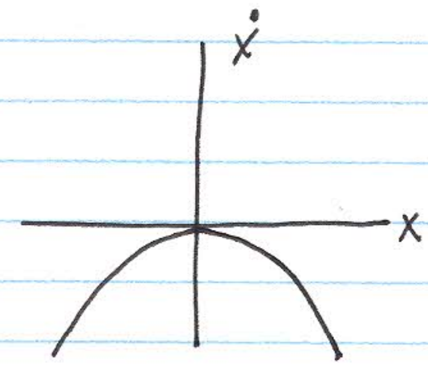
$p = (0,0)$

## Transcritical bifurcation

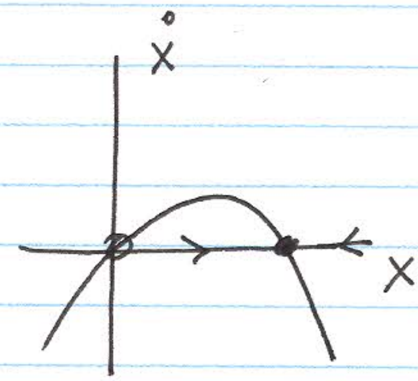
$\dot{x} = \varepsilon x - x^2$




$\varepsilon < 0$

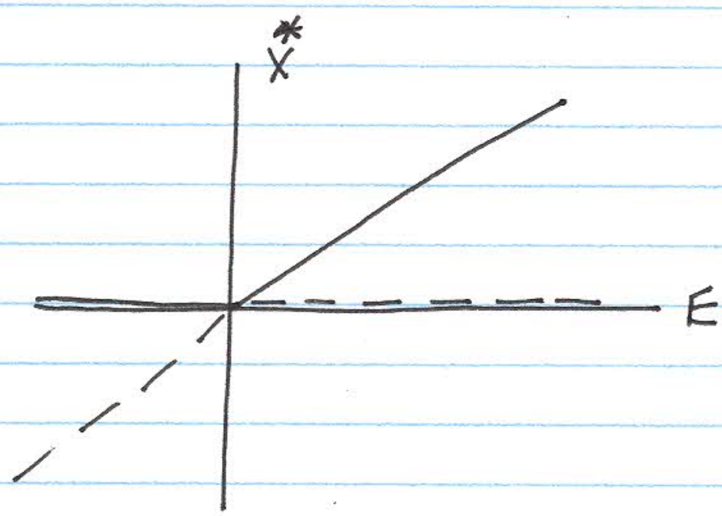



$\varepsilon = 0$




$\varepsilon > 0$

$\frac{\partial f}{\partial x} = \varepsilon - 2x$ $\qquad$ $0 = f(x^*; \varepsilon) \Rightarrow x^* = 0$ or $x^* = \varepsilon$

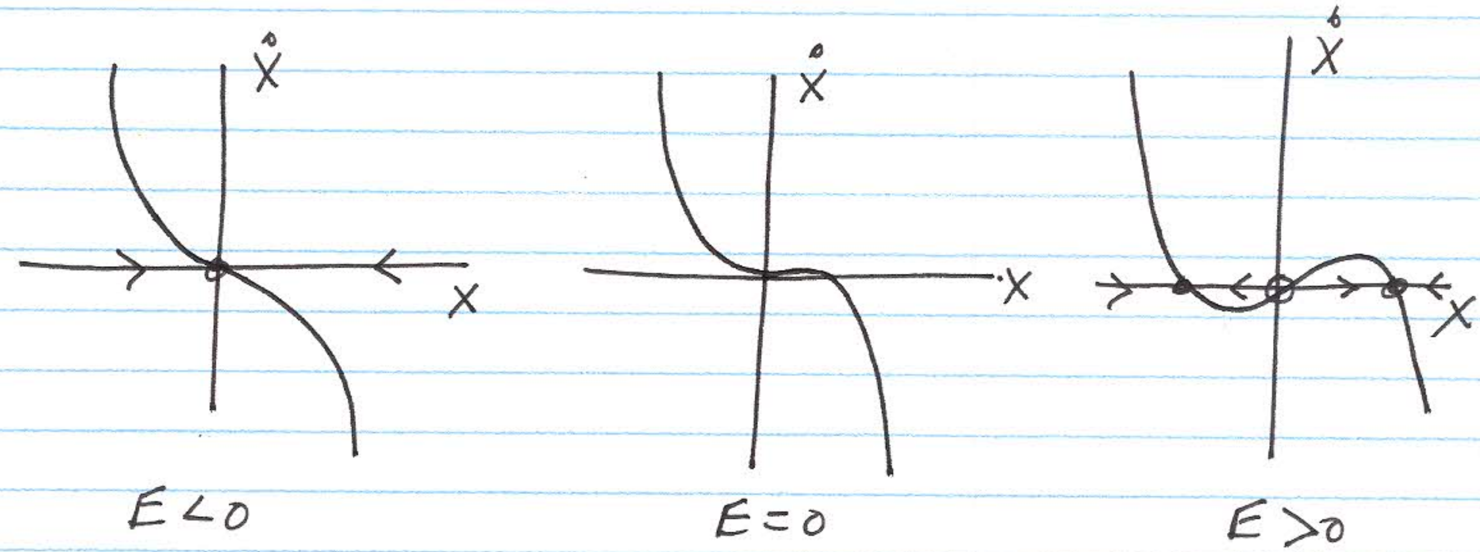



Bifurcation at

$P = (0,0)$

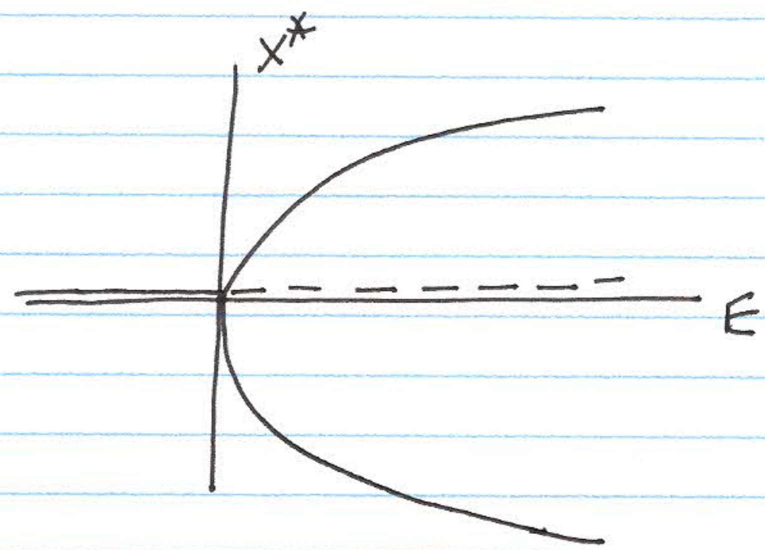
## Pitchfork bifurcation

$$\dot{x} = Ex - x^3$$




$$\frac{\partial f}{\partial x} = E - 3x^2 \qquad 0 = f(x^*, E) \Rightarrow x^* = 0 \quad \underline{\underline{or}}$$

$$x^* = \pm\sqrt{E}$$

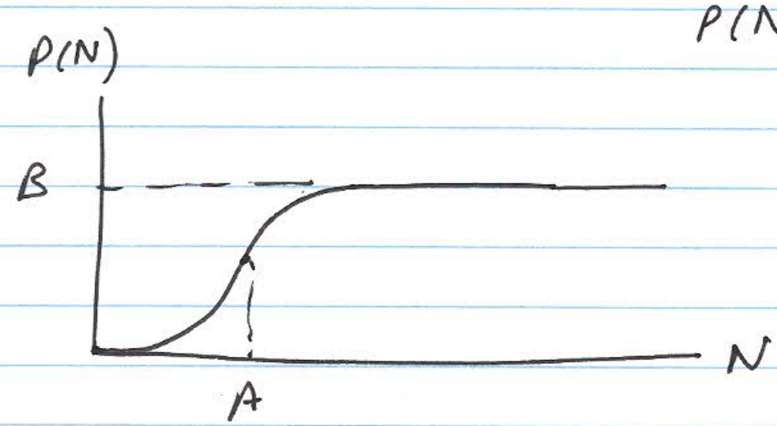



Bifurcation at

$P=(0,0)$

Spruce Budworm model Ludwig (1978)

$$\dot{N} = \underbrace{RN\left(1-\frac{N}{K}\right)}_{\text{logistic growth}} - \underbrace{\frac{BN^2}{A^2+N^2}}_{\substack{\text{predation}\\ \text{by birds}\\ P(N)}} = f(N)$$




"switching"

Under what conditions does an outbreak occur?

Rescale - nondimensionalize.

Eg 1 $\quad \dfrac{dN}{dt} = rN\left(1-\dfrac{N}{K}\right)$ $\quad N_0$ given

$N \sim \#$ $\quad t \sim$ time $\qquad$ variables units must match across $+, -, =$

so... $K \sim \#$ $\quad r \sim 1/\text{time}$. parameters

rescale dimensional variables by parameters to create dimensionless variables

"if you use it you lose it.."

$X = N/K \qquad \tau = rt$

$$\frac{dx}{d\tau} = \frac{1}{rK}\frac{dN}{dt} = \frac{1}{rK} rN\left(1-\frac{N}{K}\right) = x(1-x)$$

$$\frac{dx}{d\tau} = x(1-x) \qquad x_0 = N_0/K$$

Any solution to this system can be mapped back to the original system by $N = xK$, $t = \tau/r$

Returning to the example at hand...

$N \sim \#$ $t \sim$ time variables

$R \sim 1/\text{time}$ $K \sim \#$ $A \sim \#$ $B \sim \#/\text{time}$

choose $x = N/A$ $\tau = Bt/A$

$$\frac{dx}{d\tau} = \frac{1}{A} \cdot \frac{A}{B} \frac{dN}{dt} = \frac{R}{B} N\left(1 - \frac{N}{K}\right) - \frac{N^2}{A^2+N^2}$$

$$= \frac{R}{B} Ax\left(1 - \frac{Ax}{K}\right) - \frac{(Ax)^2}{A^2+(Ax)^2}$$

$$= \left(\frac{RA}{B}\right) x\left(1 - \frac{x}{K/A}\right) - \frac{x^2}{1+x^2}$$

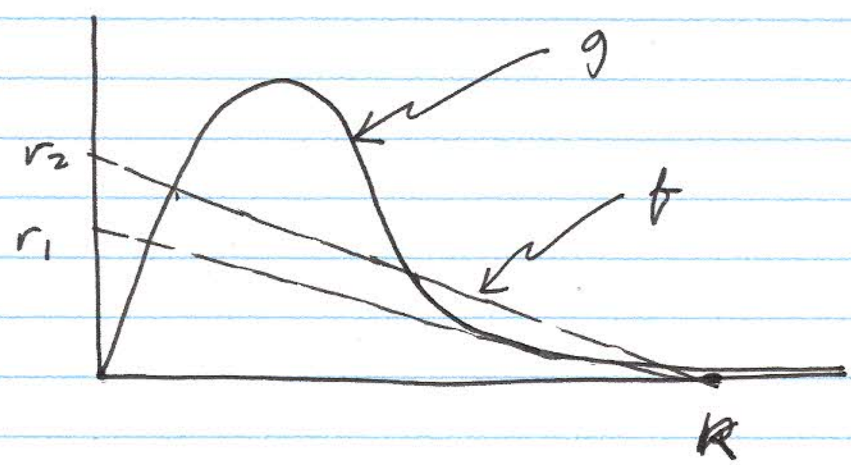
$$= r x\left(1 - \frac{x}{k}\right) - \frac{x^2}{1+x^2}$$ where...

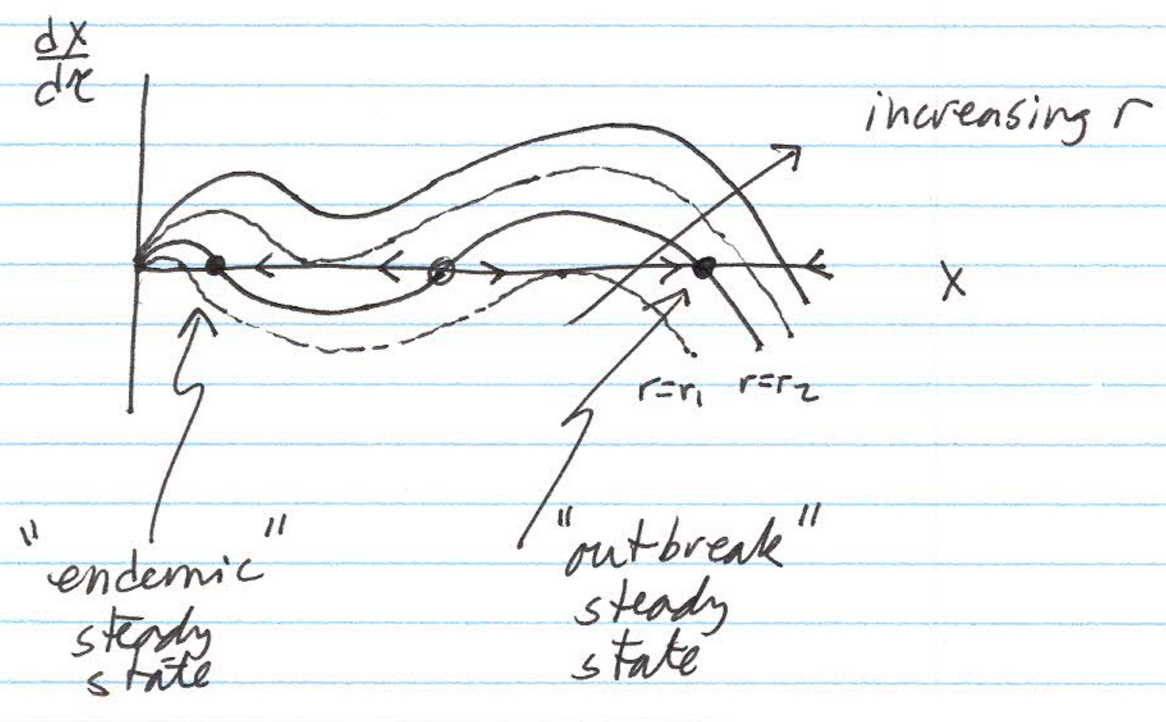
$r = \frac{RA}{B}$ $k = K/A$

$$\frac{dx}{d\tau} = \overbrace{r x \left(1 - \frac{x}{k}\right)}^{x f(x)} - \overbrace{\frac{x^2}{1+x^2}}^{x g(x)} = h(x)$$

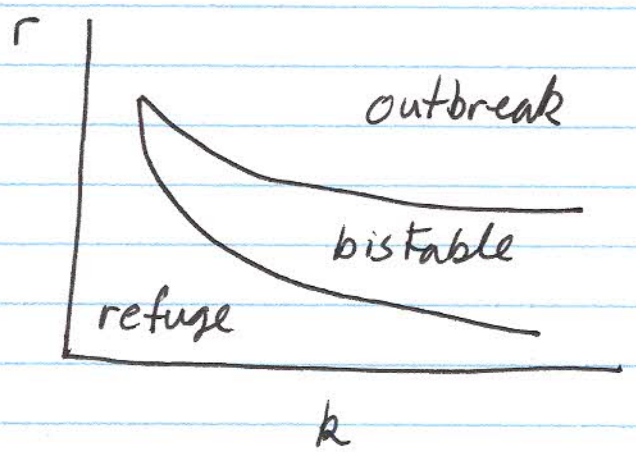
steady states : $x = 0$ or

$$\underbrace{r\left(1 - \frac{x}{k}\right)}_{f} = \underbrace{\frac{x}{1+x^2}}_{g}$$



- large $r$ means only stable non-negative steady state is outbreak
- small $r$ means only stable non-negative steady state is endemic
- intermediate $r$ means coexistence of endemic and outbreak steady states
- two saddle-node bifurcations!




See Strogatz Nonlinear Dynamics & Chaos (1994)

also

Recall $r = AR/B$ $\quad k = K/A$

How to interpret ecologically?